Average Energy Stored in a Mutual Inductor at AC Steady State

Onur Memioğlu

April 22, 2013

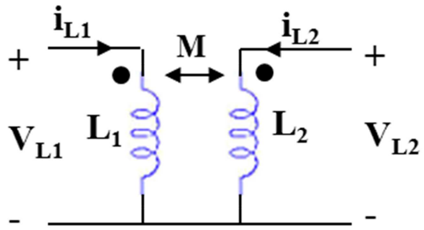
Abstract: The formula for the average stored energy in mutual inductors under AC excitation is derived in two different ways. [1]

First Method:

Firstly, we compute the instantaneous power of a mutual inductor. Instantaneous power of a component is equal to:

$$p(t) = i(t).v(t)$$

For mutual inductor:



$$p(t) = i(t).v(t)$$

$$p(\mathrm{t}) = i_1(\mathrm{t}).v_1(\mathrm{t}) + i_2(\mathrm{t}).v_2(\mathrm{t})$$

$$p(t) = \begin{pmatrix} i_1(t) & i_2(t) \end{pmatrix}.\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$$

By mutual inductance equation:

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix} . \begin{pmatrix} {di_1(t)}/{dt} \\ {di_2(t)}/{dt} \end{pmatrix}$$

[1] Notes are prepared to assist EE 202 course of Department of Electrical-Electronics Engineering, Middle East Technical University (METU), Ankara, Turkey. The author is a second year undergraduate student taking the course in Spring 2012-13 semester.

p(t) becomes:

$$p(t) = \begin{pmatrix} i_1(t) & i_2(t) \end{pmatrix} . \begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix} . \begin{pmatrix} di_1(t)/_{dt} \\ di_2(t)/_{dt} \end{pmatrix}$$

$$p(t) = \begin{pmatrix} i_1(t).L_1 + M.i_2(t) & i_1(t).M + i_2(t).L_2 \end{pmatrix} . \begin{pmatrix} di_1(t)/_{dt} \\ di_2(t)/_{dt} \end{pmatrix}$$

$$p(t) = i_1(t).L_1.\frac{di_1(t)}{dt} + M.i_2(t).\frac{di_1(t)}{dt} + i_1(t).M.\frac{di_2(t)}{dt} + i_2(t).L_2.\frac{di_2(t)}{dt}$$

The terms that include M can be arranged as

$$M.i_2(t).\frac{di_1(t)}{dt} + i_1(t).M.\frac{di_2(t)}{dt} = M.\frac{d(i_1(t).i_2(t))}{dt}$$

(Multiplication rule of the derivative)

Therefore p(t) becomes:

$$p(t) = i_1(t).L_1.\frac{di_1(t)}{dt} + M.\frac{d(i_1(t).i_2(t))}{dt} + i_2(t).L_2.\frac{di_2(t)}{dt}$$

From this equation total energy can be computed by integrating the instantaneous power from -∞ to an arbitrary time t.

$$E(t) = \int_{-\infty}^{t} P(t').dt' = \int_{-\infty}^{t} (i_1(t').L_1.\frac{di_1(t')}{dt'} + M.\frac{d(i_1(t').i_2(t'))}{dt'} + i_2(t').L_2.\frac{di_2(t')}{dt'}).dt'$$

dt' terms cancel inside the integration:

$$E(t) = \int_{-\infty}^{t} i_1(t').L_1.di_1(t') + \int_{-\infty}^{t} i_2(t').L_2.di_2(t') + \int_{-\infty}^{t} M.d(i_1(t').i_2(t'))$$

At -∞, $i_1(t)$ and $i_2(t)$ are equal to zero. Therefore $E(t)$ becomes:

$$E(t) = \frac{L_1.{i_1}^2(t)}{2} + \frac{L_2.{i_2}^2(t)}{2} + M.i_1(t).i_2(t)$$

Average stored energy in the mutual inductor becomes:

$$\bar{E} = \frac{1}{T}\int_0^T E(t).dt = \frac{1}{T}\int_0^T \left(\frac{L_1.{i_1}^2(t)}{2} + \frac{L_2.{i_2}^2(t)}{2} + M.i_1(t).i_2(t)\right).dt$$

$$= \frac{1}{T}(\int_0^T \frac{L_1.{i_1}^2(t)}{2}.dt + \int_0^T \frac{L_2.{i_2}^2(t)}{2}.dt + \int_0^T M.i_1(t).i_2(t).dt$$

Assume $i_1(t) = I_1.\cos(\omega t + \theta_1)$

$i_2(t) = I_2.\cos(\omega t + \theta_2)$

$$= \frac{L_1}{2}.\left(\sqrt{\frac{1}{T}\int_0^T {i_1}^2(t).dt}\right)^2 + \frac{L_2}{2}.\left(\sqrt{\frac{1}{T}\int_0^T {i_2}^2(t).dt}\right)^2$$

$$+ \frac{1}{T}\int_0^T M.I_1.I_2.\cos(\omega t + \theta_1).\cos(\omega t + \theta_2).dt$$

The terms in square root are equal to the RMS values of the inductor branch currents. For the third integral, we can convert multiplication of two cosines into addition of two cosines by the following formula:

$$\cos(a).\cos(b) = \frac{1}{2}.(\cos(a+b) + \cos(a-b))$$

The average stored energy becomes:

$$= \frac{L_1}{2}.I_{1,RMS}^2 + \frac{L_2}{2}.I_{2,RMS}^2 + \frac{M.I_1.I_2}{2.T}.\int_0^T (\cos(2\omega t + \theta_1 + \theta_2) + \cos(\theta_1 - \theta_2)).dt$$

Integration of $\cos(2\omega t + \theta_1 + \theta_2)$ over the whole period is zero because cosine is a periodic function. The only remaining term is $\cos(\theta_1 - \theta_2)$.

$$\bar{E} = \frac{L_1}{2}.I_{1,RMS}^2 + \frac{L_2}{2}.I_{2,RMS}^2 + \frac{M.I_1.I_2}{2.T}.\int_0^T \cos(\theta_1 - \theta_2).dt$$

$$\bar{E} = \frac{L_1}{2}.I_{1,RMS}^2 + \frac{L_2}{2}.I_{2,RMS}^2 + \frac{M.I_1.I_2}{2.T}.\cos(\theta_1 - \theta_2).T$$

$$\bar{E} = \frac{L_1}{2}.I_{1,RMS}^2 + \frac{L_2}{2}.I_{2,RMS}^2 + \frac{M.I_1.I_2}{2}.\cos(\theta_1 - \theta_2)$$

$$\bar{E} = \frac{L_1}{2}.I_{1,RMS}^2 + \frac{L_2}{2}.I_{2,RMS}^2 + M.I_{1,RMS}.I_{2,RMS}.\cos(\theta_1 - \theta_2)$$

In the end result we can see that the average energy stored in the mutual inductor at the AC steady state depends only on the RMS values of the currents and the cosine of the relative phase between the currents. If the phase difference is equal to zero then the relation simply drops to:

$$\bar{E} = \frac{L_1}{2}.I_{1,RMS}^2 + \frac{L_2}{2}.I_{2,RMS}^2 + M.I_{1,RMS}.I_{2,RMS}$$

Second Method:

By computing the complex power of the system:

$$S = P + jQ$$

Since, $Q_{mutual\ inductor} = 2.\omega.E_{avg}$

By finding the Q in the complex power, we can eventually solve for the average energy stored in the mutual inductor.

For any component power can be written as:

$$S = P + jQ = \frac{1}{2}.I^*.V$$

in phasor domain. (note that *V* and *I* are voltage and current phasors, respectively; and star (*) indicates the conjugate of the current)

From the mutual inductor equation:

$$P + jQ = \frac{1}{2}.\begin{pmatrix} I^*{}_1 & I^*{}_2 \end{pmatrix}.j.\omega.\begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix}.\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$P + jQ = \frac{1}{2}.j.\omega.\begin{pmatrix} I^*{}_1.L_1 + M.I^*{}_2 & I^*{}_1.M + I^*{}_2.L_2 \end{pmatrix}.\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$P + jQ = \frac{1}{2}.j.\omega.(I_1.I^*{}_1.L_1 + M.I^*{}_1.I_2 + I_1.I^*{}_2.M + I_2.I^*{}_2.L_2)$$

$$I_1.I^*{}_1 = |I_1|^2$$

$$I_2.I^*{}_2 = |I_2|^2$$

$$I_1.I^*{}_2 = |I_1|.|I_2|.e^{j(\theta_1 - \theta_2)}$$

$$I_2.I^*{}_1 = |I_1|.|I_2|.e^{j(\theta_2 - \theta_1)}$$

By inserting these values we obtain:

$$P + jQ = \frac{1}{2}.j.\omega.(|I_1|^{\,2}.L_1 + M.|I_1|.|I_2|.e^{j(\theta_1-\theta_2)} + M.|I_1|.|I_2|.e^{j(\theta_2-\theta_1)} + |I_2|^{\,2}.L_2)$$

The terms that have M can be arranged as:

$$M.|I_1|.|I_2|.(e^{j.(\theta_1-\theta_2)} + e^{-j.(\theta_1-\theta_2)})$$

Sum of the exponential terms is equal to:

$$\left(e^{j.(\theta_1-\theta_2)} + e^{-j.(\theta_1-\theta_2)}\right) = 2.\cos(\theta_1 - \theta_2)$$

By combining these values:

$$P + jQ = \frac{1}{2}.j.\omega.(|I_1|^{\,2}.L_1 + 2.M.|I_1|.|I_2|.cos(\theta_1 - \theta_2) + |I_2|^{\,2}.L_2)$$

As we can see, the right hand side does not have a real component. Therefore, P is equal to zero.

$$jQ = \frac{1}{2}.j.\omega.(|I_1|^{\,2}.L_1 + 2.M.|I_1|.|I_2|.cos(\theta_1 - \theta_2) + |I_2|^{\,2}.L_2)$$

$$\frac{Q}{2.\omega} = \frac{1}{2}.\frac{1}{2}.(|I_1|^{\,2}.L_1 + 2.M.|I_1|.|I_2|.cos(\theta_1 - \theta_2) + |I_2|^{\,2}.L_2)$$

By distributing one of the ½ coefficients inside the equation, we obtain RMS values for each current component. Then,

$$\frac{Q}{2.\omega} = \frac{1}{2}.(\,{I_{1,RMS}}^2.L_1 + 2.M.I_{1,RMS}.I_{2,RMS}.cos(\theta_1 - \theta_2) + {I_{2,RMS}}^{\,2}.L_2)$$

From, $Q_{mutual\ inductor} = 2.\omega.E_{avg}$

$$E_{avg} = \frac{Q}{2.\omega} = \frac{1}{2}.(\,{I_{1,RMS}}^2.L_1 + {I_{2,RMS}}^{\,2}.L_2 + 2.M.I_{1,RMS}.I_{2,RMS}.cos(\theta_1 - \theta_2))$$

Hence we obtained the same result with the first method:

$$\bar{E} = \frac{L_1}{2}.I_{1,RMS}^2 + \frac{L_2}{2}.I_{2,RMS}^2 + M.I_{1,RMS}.I_{2,RMS}.\cos(\theta_1 - \theta_2)$$